Mean field Theory of Quantum Phase Transition in the 1d Transverse field Ising Model.

The actual Hamiltonian is :

$$H = -J \sum_i \sigma_i^z \sigma_{i+1}^z - h \sum_i \sigma_i^x$$

Within the mean-field theory, we replace the interaction term (i.e. the J term) as.

$$\sigma_i^z \sigma_{i+1}^z \rightarrow \langle \sigma_i^z \rangle \sigma_{i+1}^z + \sigma_i^z \langle \sigma_{i+1}^z \rangle - \langle \sigma_i^z \rangle \langle \sigma_{i+1}^z \rangle$$

Further we assume a <u>homogeneous</u> solution $\langle \sigma_i^z \rangle = m \quad \forall i$.

The mean-field Hamiltonian becomes

$$H_{MF} = -2Jm\sum_i \sigma_i^z - h\sum_i \sigma_i^x + NJm^2$$

Were the coordination no. z (here z = 2), we would have

$$H_{MF} = -zJm\sum_i \sigma_i^z - h\sum_i \sigma_i^x + \frac{Nz}{2}Jm^2$$

The ground state energy is

$$\frac{E}{N} = -\sqrt{(zJm)^2 + h^2} + \frac{Jzm^2}{2}$$

Doing a taylor expansion in m, one obtains,

$$\frac{E}{N} = -h \left[1 + \left(\frac{z J m}{h} \right)^2 \right]^{1/2} + \frac{J z m^2}{2}$$

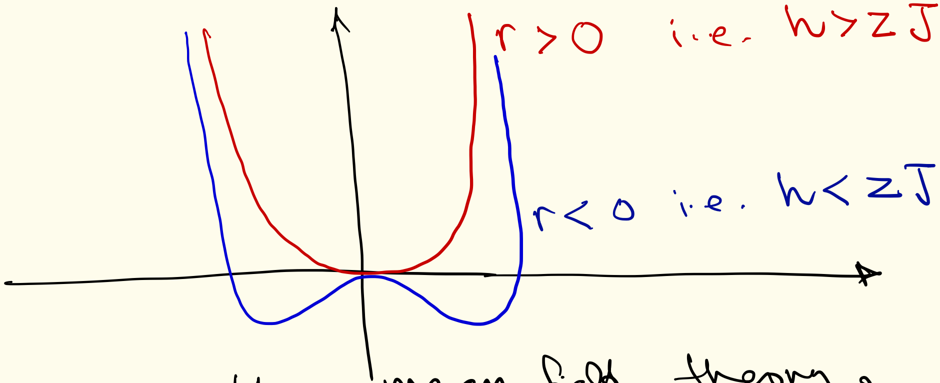
$$\simeq -h \left[1 + \frac{1}{2} \frac{(z J m)^2}{h^2} - \frac{1}{4} \left(\frac{z J m}{h} \right)^4 + \cdots \right] + \frac{J z m^2}{2}$$

$$= r m^2 + u m^4 + \text{constant}$$

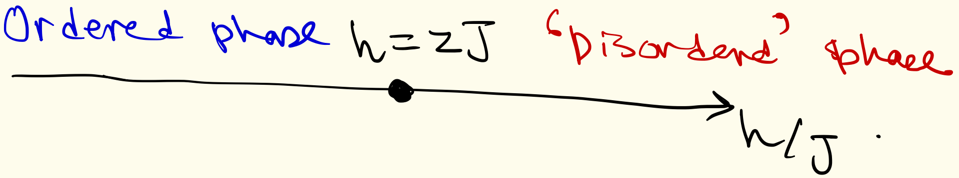
where $$r = \frac{J z}{2} - \frac{(z J)^2}{2 h}$$

$$u = \frac{1}{4} \frac{(z J m)^4}{h^3}$$

This has exactly the same form as the Landau free energy for the Ising transition!



Thus, within the mean-field theory, the sym. is spontaneously broken when $h < zJ$.



An alternate method is to find m self consistently,

$$m = \langle \psi_0 | \sigma_i^z | \psi_0 \rangle$$ where

$|\psi_0\rangle$ is the mean-field ground state.

$$\langle \psi_0 | \sigma_i^z | \psi_0 \rangle = \frac{zJm}{\sqrt{(zJm)^2 + h^2}}$$

$$\Rightarrow \quad m = \frac{zJm}{\sqrt{(zJm)^2 + h^2}}$$

$$\Rightarrow (zJm)^2 + h^2 = (zJ)^2$$

$$\Rightarrow \quad m^2 = \frac{(zJ)^2 - h^2}{(zJ)^2}$$

$$\Rightarrow \quad m = \pm \sqrt{1 - \frac{h^2}{(zJ)^2}}$$

Thus, m vanishes above the critical ratio: $\left(\frac{h}{J}\right)_c = z$, as also seen above using the Landau theory.